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NONLINEAR OSCILLATIONS OF VISCO-ELASTIC PLATES

by

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a typical length — Equation (4)

a^i components of acceleration vector — Equation (45)

b_i constants — Equation (10)

C boundary curve

D_i constants

$D = Eh^3/12(1-\nu^2)$ Plate bending rigidity

E_z Young's modulus in z-direction — Equation (27)

E, E' Young's Modulus

E_* operator used in place of E for internal damping — Equation (9)

e small perturbation parameter

e_{ij} deviatric part of strain tensor — Equations (23,24)

$\bar{F}(t)$ Laplace transform of F(t) — Equation (12)

F^i or F_x, F_y, F_z body force components per unit mass — Equation (34)

$F_o(\xi,\eta,t)$ Airy stress function — Equation (71)

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G, G' shear modulus

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V strain energy Equation (83)

x^i or x,y,z rectangular cartesian coordinates

y^i curvilinear coordinates in deformed state

α,γ positive numbers Equation (29)

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$\delta_{ij}=\delta_i^{\,j}=\delta^{ij}$ Kronecker deltas

$\delta(t)$ Dirac-delta function Equation (25)

Δ Laplacian operator Equation (54)

Δ',Δ^* Differential operators Equation (67)

ε_{ij} covariant components of strain tensor Equations (2,5)

ζ non-dimensional transverse coordinate

θ,θ_2,θ_3 strain invariants Equations (6,48,86)

θ_o dilatation Equation (64)

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κ constant Equation (37)

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ν Poisson's ratio

ξ,η,ζ non-dimensional coordinates Equation (4)

ρ mass density per unit volume Equation (39)

σ_{ij} covariant components of stress tensor referred to curvilinear coordinates

Σ edge surface

$\phi(t)$ function of time — Equation (17)

ω_o rotation — Equation (64)

$[\gamma,\phi]$ operator of two functions — Equation (59)

$*$, asterisk subscript in elastic coefficients; denotes convolution — Equation (17)

$u_{ɔ,\xi} = \partial u_{ɔ}/\partial\xi$

$O(e^k)$ terms of "k" th order in e

$[f]_e$ indicates only the elastic part of f

Abstract

By means of a systematic approximation based on the perturbation theory, from the three-dimensional, non-linear elasticity theory, five partial differential equations are obtained for the non-linear oscillations of visco-elastic plates. The plate can undergo a finite transverse deflection. The rotatory inertia, shear deformation and effect of body force and moment components are taken into account. The present theory is the generalization of Föppl-Kármán-Timoshenko theory of large static deflection on the one hand and Uflyand-Mindlin theory of small oscillation theory on the other hand.

Boundary and initial conditions are discussed and an expression for the strain energy is given.

1. Introduction

Föppl obtained two simultaneous non-linear partial differential equations to study the large static deflection of plates [1]. Von Kármán and S. Timoshenko reformulated and studied these equations for various plate problems. Many other authors used these equations, now known as Föppl-Kármán-Timoshenko theory. Assumptions of this theory are not systematic and thus it is desirable to know whether there are other terms missing from these equations which are of the same order of magnitude with the present ones. Also for dynamic problems the inertia terms must be taken into account.

The effect of shear deformation in static plate problems was first considered by E. Reissner [2]. Uflyand [3] and Mindlin [4] have reformulated small oscillation problems of plates and took into account the effect of rotatory inertia and shear deformation.

Recently for the general formulation of shell problems, many papers have appeared, most of which are in tensor notation or written in intrinsic coordinates which do not lend themselves easily to application [5, 6, 7].

The highly rigorous nature of these papers leads to very high complexity, thus making the solution of even very simple problems prohibitive or impossible. The use of the highly productive method of engineers, the intuition, is thereby hindered.

In impact problems, where the transient stage is important, a further complication must be included into the formulation, namely the internal damping effect.

With these views in mind, the reformulation of the plate theory is made. Consequently, the present theory includes the effects of: (a) finite transverse deflection, (b) shear deformation, (c) rotatory inertia, (d) internal damping, (e) body force and moments. The theory is based on the following assumptions:

1. The deformation components are expandable into power series of non-dimensional transverse coordinates ζ times small perturbation parameter e which is chosen as the non-dimensional thickness of the plate (see equation (3) below).

2. Hooke's Law is valid between stress and strain components.

3. The coefficient of shear viscosity, at the median plane, is a constant multiple of the coefficient of normal viscosity at the same plane.

The first assumption is similar to Goodier's [8] assumption in treating problems of beam and plate. He assumes that the components of the stress tensor are developable into power series of non-dimensional thickness parameter e. In treating the finite deformation problems, this method becomes very complicated and cumbersome. It seems that the same purpose can be accomplished directly by dealing with the deformation components. We thus start from a point at which he ends up. The advantage of this procedure is that it avoids integration of some systems of differential equations which may be quite complicated in a finite deformation theory. Difficulty arises however in obtaining desirable orders of magnitude for transverse stress components for the isotropic material. While this point is easily explained for an orthotropic material, it cannot be explained for an isotropic material without considering the higher order terms in the expressions of the deformation components. Such an effort seems to be hardly worth the trouble.

The second assumption is controversial but a commonly used one.

The last assumption need not be included into the theory. The whole theory can be carried out without using this assumption. However, unnecessary complications are avoided by using it.

The present theory ends after obtaining differential equations of the lowest order pertinent functions in the expansion of deformation components.

Five partial differential equations are obtained whose solution is believed to explain the behavior of finite oscillations of visco-elastic plates to a reasonable degree of approximation. These equations are non-linear, and contain all previous technical plate theories as special cases.

Boundary and initial conditions are discussed. It is found that five conditions are needed for every part of the edge surface which has continuous normals possessing first order continuous derivatives.

Finally, an expression for the strain energy is given which consists of two parts, namely: Elastic energy and Energy of dissipation.

2. Components of Strain Tensor

Let x^1, x^2, x^3 or invariably x, y, z be the rectangular cartesian coordinates of a point of the plate before deformation. We select the (x^1, x^2)-plane as the median plane of the plate and x^3 as the direction perpendicular to this plane, so that x^1, x^2, x^3 makes a right-handed coordinate system. The planes $x^3 = \pm h/2$ will be taken as the upper and lower faces of the plate. Components of the deformation vector in rectangular cartesian coordinates will be denoted by u^i or (u, v, w) and covariant components of the strain tensor by ε_{ij} or $(\varepsilon_{xx}, \ldots, \varepsilon_{xy})$. The length element ds in the deformed body is given by [9]:

$$ds^2 = g_{ij}\, dx^i dx^j = (\delta_{ij} + 2\varepsilon_{ij})\, dx^i dx^j \tag{1}$$

$$2\varepsilon_{ij} = \frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} + \frac{\partial u^k}{\partial x^i}\frac{\partial u^k}{\partial x^j} \qquad (i,j = 1,2,3) \tag{2}$$

where:

$$\delta_{ij} = \delta_i^{\ j} = \delta^{ij} = \begin{cases} 1 \text{ when } i=j \\ o \text{ when } i \neq j \end{cases}$$

are the Kronecker deltas, and g_{ij} are the covariant components of the metric tensor for the deformed body.

Throughout the paper the usual summation convention will be used; namely, repeated indices denote summation over the range (1, 2, 3). We note that there is no need to distinguish covariant mixed and contravariant components of any tensor in cartesian coordinates. Hence $\varepsilon_{ij} = \varepsilon_i^{\ j} = \varepsilon^{ij}$, $u^i = u_i$.

We assume that $u^i(x^1, x^2, x^3, t)$ can be expanded into power series of $x^3/h = \zeta$:

$$\left.\begin{aligned} u/a &= e^2[u_o(\xi,\eta,t) + \zeta u_1(\xi,\eta,t) + e^3\zeta^3 u_3(\xi,\eta,t) + \ldots] \\ v/a &= e^2[v_o(\xi,\eta,t) + \zeta v_1(\xi,\eta,t) + e^3\zeta^3 v_3(\xi,\eta,t) + \ldots] \\ w/a &= e[w_o(\xi,\eta,t) + e^2\zeta^2 w_2(\xi,\eta,t) + e^4\zeta^4 w_4(\xi,\eta,t) + \ldots] \end{aligned}\right\} \tag{3}$$

where ξ, η, ζ are non-dimensional coordinates, a is a typical length and e is a small parameter.

$$x/a = \xi\,,\ y/a = \eta\,,\ z/a = e\zeta\,,\ e = h/a\,,\ e \ll 1 \tag{4}$$

The explicit forms of the strain components are obtained by substituting equations (3) into equations (2). Hence:

$$\left.\begin{aligned}
\varepsilon_{xx} &= e^2[u_{o,\xi} + \tfrac{1}{2} w_{o,\xi}{}^2 + \zeta u_{1,\xi}] + O(e^4) \\
\varepsilon_{yy} &= e^2[v_{o,\eta} + \tfrac{1}{2} w_{o,\eta}{}^2 + \zeta v_{1,\eta}] + O(e^4) \\
\varepsilon_{zz} &= e^2[\tfrac{1}{2}(u_1{}^2 + v_1{}^2) + 2\zeta w_2] + O(e^4) \\
\varepsilon_{xy} &= \tfrac{1}{2}e^2[u_{o,\eta} + v_{o,\xi} + w_{o,\xi}w_{o,\eta} + \zeta(u_{1,\eta} + v_{1,\xi})] + O(e^4) \\
\varepsilon_{yz} &= \tfrac{1}{2} e(v_1 + w_{o,\eta}) + O(e^3) \\
\varepsilon_{zx} &= \tfrac{1}{2} e(u_1 + w_{o,\xi}) + O(e^3)
\end{aligned}\right\} \quad (5)$$

where indices after a comma represent differentiation, i.e. $u_{o,\xi} = \partial u_o/\partial\xi$ etc. In equation (5) the second order terms represent the effect of large deformations which would be encountered in a second order theory. The second order terms occuring in the expressions of ε_{xx}, ε_{yy}, and ε_{xy} are identical to those of Föppl-Kármán-Timoshenko theory of large static deflection. The classical plate theory assumes $u_1 = -w_{o,\xi}$, $v_1 = -w_{o,\eta}$, hence $(\varepsilon_{yz}, \varepsilon_{zx}) = O(e^3)$. Consequently the transverse shear strains ε_{yz}, ε_{zx}, as well as ε_{zz}, are taken as zero. The present theory will not ignore these strain components.

3. Stress-Strain Relations for an Elastic Medium with Internal Damping

The literature contains various attempts to generalize the classical stress-strain law of elasticity so as to include the effect of internal damping. Early discoveries are due to O. Meyer [10], W. Voigt [11] and K. Sezawa [12] which can, in the modern notation, be expressed as:

$$\left.\begin{aligned} \sigma_{ij} &= (\lambda + \lambda' \frac{\partial}{\partial t})\,\theta\,\delta_{ij} + (2\mu + 2\mu' \frac{\partial}{\partial t})\,\varepsilon_{ij} \\ \theta &= \delta^{ij}\varepsilon_{ij} = \varepsilon^{i}{}_{i} \end{aligned}\right\} \quad (6)$$

where λ and μ are the classical Lamé constants and λ' and μ' are the normal and shear solid viscosities. Here σ_{ij} are the mixed components of the stress tensor which are referred to the curvilinear coordinates y^i of the deformed state, while strain components are referred to the cartesian coordinates x^i of the undeformed state.

Coordinates y^1 and y^2 are perpendicular to each other and lie in the tangent plane of the median plane and y^3 is perpendicular to this plane and makes a right hand system with y^1 and y^2.

The controversial assumption underlying equations (6) seems to be as good as any one in use relating stress components to strain components.

Coefficients λ, μ, are related to Young's modulus E, Shear modulus G and Poisson's ratio ν by the following relations:

$$\lambda = \nu E/(1-2\nu)(1+\nu) \quad , \quad \mu = G \tag{7}$$

In a similar manner E′, G′ and ν can be defined by:

$$\lambda' = \nu E'/(1-2\nu)\ (1+\nu) \quad , \quad \mu' = G' \tag{8}$$

Consequently, if one defines:

$$\left.\begin{aligned} \underset{*}{\lambda} &= \lambda + \lambda'\frac{\partial}{\partial t} \quad , \quad \underset{*}{\mu} = \mu + \mu'\frac{\partial}{\partial t} \\ \underset{*}{E} &= E + E'\frac{\partial}{\partial t} \quad , \quad \underset{*}{G} = G + G'\frac{\partial}{\partial t} \end{aligned}\right\} \tag{9}$$

equations (6) take the same form as those of the classical elasticity theory, which further leads to a considerable simplification.

We note that the strain components in equations (6) are expressed in cartesian coordinates. Hence covariant, contravariant and mixed components of stress tensor need not be distinguished for the present approximation.

When a Voigt solid is subject to oscillating stresses, the rate of dissipation of energy is proportional to the square of the frequency of oscillation, as may be observed from equation (6) for the one-dimensional case. Lord Kelvin [13] has observed that the rate of dissipation of energy increases less rapidly than the square of the frequency. This led to the generalization of Poynting and Thomson [14] and more recently to that of Alfrey [15, 16], which contains all other theories given before as a special case.(+) Alfrey's stress-

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(+) L. Boltzmann uses the superposition principle which seems to be less suitable for the purpose of generalization.

strain relations for an incompressible material are expressed in the following form:

$$\left.\begin{aligned} &P\sigma_{ik} = P\,\sigma\,\delta_{ik} + 2Q\varepsilon_{ik}\,, \qquad \sigma = \tfrac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \\ &\varepsilon_{ii} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0 \\ &P = \sum_{i=o}^{m} a_i \frac{\partial^i}{\partial t^i}\,, \quad Q = \sum_{i=o}^{n} b_i \frac{\partial^i}{\partial t^i} \end{aligned}\right\} \tag{10}$$

where conatants a_i and b_i are to be determined from the experiment.

We now generalize this equation further to include normal viscosities in the case of compressible material, and write:

$$\left.\begin{aligned} &P\sigma_{ik} = R\theta\delta_{ik} + 2Q\varepsilon_{ik} \\ &R = \sum_{i=o}^{\ell} c_i \frac{\partial^i}{\partial t^i}\,, \ \theta = \varepsilon_{ii} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \end{aligned}\right\} \tag{11}$$

Here P and Q are the same as in equation (10). We now solve σ_{ik} from differential equation (11) by using operational methods.

Let $\bar{F}(s)$ be the Laplace transform of F(t) as defined by:

$$\bar{F}(s) = \int_0^{\infty} e^{-st} F(t)\,dt \tag{12}$$

Applying the Laplace transform to the expression of $P\sigma_{ik}$ in equation (11) (i.e. by taking $P\sigma_{ik}$ for F(t) in equation (12)), we obtain

$$\bar{\sigma}_{ik} = \bar{L}(s)\,\bar{\theta}\,(x_1, x_2, x_3, s)\,\delta_{ij} + 2\bar{M}(s)$$

$$\bar{\varepsilon}_{ij}(x_1, x_2, x_3, s) + \bar{N}(x_1, x_2, x_3, s) \tag{13}$$

where:

$$\left.\begin{aligned}\bar{L}(s) &= \Big(\sum_{i=o}^{\ell} c_i s^i\Big)/\Big(\sum_{i=o}^{m} a_i s^i\Big), \quad \bar{M}(s) = \Big(\sum_{i=o}^{n} b_i s^i\Big)/\Big(\sum_{i=o}^{m} a_i s^i\Big) \\ \bar{N}(s) &= \Big(\sum_{i=o}^{m-1} D_i s^i\Big)/\Big(\sum_{i=o}^{m} a_i s^i\Big), \quad D_i = D_i(x_1, x_2, x_3)\end{aligned}\right\} \tag{14}$$

Taking the inverse transform of equation (13) we obtain

$$\sigma_{ij} = \delta_{ij}\int_o^t L_o(t-\tau).\ \theta(x_1, x_2, x_3, \tau)\,d\tau$$

$$+ 2\int_o^t M_o(t-\tau)\ \varepsilon_{ij}(x_1, x_2, x_3, \tau)\,d\tau + N_o(x_1, x_2, x_3, t) \tag{15}$$

If the system has no initial stress then σ_{ij} must vanish with ε_{ij}, thus giving $N_o \equiv 0$. Furthermore, writing the elastic parts separately in equation (15) we have:

$$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij} + \delta_{ij}\int_o^t L(t-\tau)\ \theta(x_1, x_2, x_3, \tau)\,d\tau$$

$$+ 2\int_o^t M(t-\tau)\varepsilon_{ij}(x_1, x_2, x_3, \tau)\,d\tau \tag{16}$$

where:

$$L(t) = L_o(t) - \lambda \qquad M(t) = M_o(t) - \mu \tag{17}$$

Thus, we arrived at V. Volterra's [17] stress-strain relation for a medium having heredity. Consequently, the general internal damping law stated by equation (11) is identical to hereditary damping which is frequently treated as a different and rather more general phenomenon. Equation (16) is, of course, identical to those used by E. Volterra [18] for an elastic medium with hereditary damping.

A useful form of the stress-strain relations (16) is obtained if we use the convolution multiplication defined by:

$$F * \phi = \int_0^t F(t-\tau)\phi(\tau)d\tau \tag{18}$$

We note that $F * \phi = \phi * F$. Equation (16) can now be written simply:

$$\sigma_{ij} = \underset{*}{\lambda}\, \theta\, \delta_{ij} + 2\underset{*}{\mu}\, \varepsilon_{ij} \tag{19}$$

where the operators $\underset{*}{\lambda}$ and $\underset{*}{\mu}$ are defined by:

$$\underset{*}{\lambda} = \lambda + L* \qquad \underset{*}{\mu} = \mu + M* \tag{20}$$

Equation (19) has exactly the same form of elastic stress-strain relations for a homogeneous, isotropic media. Therefore, substitution of $\underset{*}{\lambda}$ and $\underset{*}{\mu}$ for λ and μ of classical elasticity theory in all equations of elasto-dynamics, gives the corresponding equations for a medium with internal damping. Attention must be given to the fact, however, that $\underset{*}{\lambda}$ and $\underset{*}{\mu}$ have time derivatives.

A more compact relation would be obtained if we eliminated θ in equation (19). Writing $i = j$ we obtain:

$$\sigma = (\underset{*}{\lambda} + \frac{2}{3}\underset{*}{\mu})\theta \tag{21}$$

Subtracting hydrostatic pressure $\sigma\delta_{ij}$ from equation (19), we obtain:

$$s_{ij} = 2\underset{*}{\mu}\, e_{ij} \tag{22}$$

where s_{ij} and e_{ij} are the deviatric parts of the stress and strain tensors and are defined by:

$$s_{ij} = \sigma_{ij} - \sigma\delta_{ij}, \qquad e_{ij} = \varepsilon_{ij} - \frac{\theta}{3}\delta_{ij} \tag{23}$$

It is important to note that the functions L and M of the stress-strain relations (16) may be discontinuous functions. However, the convolution integrals are continuous and have continuous derivatives with respect to time. In fact, we can obtain a general expression for L(t) and M(t) by getting the inverse transforms of $\overline{L}$ and $\overline{M}$. In the most general case we may have $m < \ell$, $m < n$, which corresponds to the case in which the highest order derivative occuring in equation (11) is that of σ_{ik}. For example, the Voight-Sezawa type of damping is of this sort. In this case, by use of the Dirac-Delta function and its derivatives, we can obtain corresponding $\underset{*}{\lambda}$ and $\underset{*}{\mu}$. If F(t) is a continuous function with continuous derivative, and F(0) = 0, integrating by parts we see that (*):

$$\frac{\partial F}{\partial t} = \int_0^t \dot{\delta}\,(t-\tau)\cdot F(\tau)d\tau \quad , \quad \dot{\delta} = \frac{\partial\delta(t)}{\partial t} \tag{24}$$

where the Dirac-Delta function $\delta(t)$ is defined by:

$$\delta(t) = \begin{cases} \infty & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases} \quad \text{and} \int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{25}$$

Hence, equation (6) is identical to equation (19) with:

$$\underset{*}{\lambda} = \lambda + \lambda'\dot{\delta}(t)* \quad , \quad \underset{*}{\mu} = \mu + \mu'\dot{\delta}(t)* \tag{26}$$

If the difference between the highest order derivatives of σ_{ij}

(*) Strictly speaking, this equation is true only in a limiting sense: $\partial F/\partial t = \lim_{\varepsilon\to 0} \int_0^{t+\varepsilon} \dot{\delta}(t-\tau,\varepsilon)f(\tau)d\tau$ where $\lim_{\varepsilon\to 0} \delta(t,\varepsilon) = \delta(t)$. See also L. Schwartz [19], vol. II. p. 17.

and ε_{ij} occuring in equation (11) is k, then L and M will contain k^{th} derivative of the delta-function(+). We can, in fact, determine and discuss all possible forms of $\underset{*}{\lambda}$ and $\underset{*}{\mu}$ by discussing the roots of the denominator of $\bar{L}$ and $\bar{M}$. We shall not, however, go into this matter here, as this would be of no interest in application for we do not know coefficients a_i, b_i and c_i. We shall be satisfied, then, by remarking that the commonly used expressions of L(t) and M(t) are of the following exponential type [18]: $\sum_{i=o}^{m} A_i e^{-\alpha_i t}$. It is easily seen that this is included in the inversion of $\bar{L}$ and $\bar{M}$. In fact, if the denominator of $\bar{L}$ and $\bar{M}$ has m real distinct root and $(\ell, n) < m$ we obtain an expression of this sort. Various attempts have been made to determine A_i and α_i in this expression [20].

It may be of further interest to note that when a_i, b_i, and c_i are functions of space coordinates (x_1, x_2, x_3) we obtain a further generalization of internal damping, leading to L and M of an inhomogeneous media which are also functions of coordinates and time.

Equation (19) also suggest that in the case of anisotropic elastic media we can use extended stress-strain relations, by simply taking the asterisk, *, as a subscript in all elastic coefficients. In the following development this will be the procedure.

(+)Mathematical justification of the use of the δ-function and its derivatives may be found in Schwartz's book [19].

4. Plate Stress-Strain Relations

We consider an orthotropic plate having a cylindrical symmetry with the axis of cylinder in the direction of the z-axis. The stress-strain relations can be written in the usual technical notation as:

$$\left.\begin{aligned}
\varepsilon_{xx} &= \frac{\sigma_{xx}}{E} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu_x}{E_z}\sigma_{zz}, \quad & \varepsilon_{xy} &= \frac{1}{2G}\sigma_{xy} \\
\varepsilon_{yy} &= -\frac{\nu}{E}\sigma_{xx} + \frac{\sigma_{yy}}{E} - \frac{\nu_x}{E_z}\sigma_{zz}, \quad & \varepsilon_{yz} &= \frac{1}{2G_z}\sigma_{yz} \\
\varepsilon_{zz} &= -\frac{\nu_z}{E}\sigma_{xx} - \frac{\nu_z}{E}\sigma_{yy} + \frac{\sigma_{zz}}{E_z}, \quad & \varepsilon_{xz} &= \frac{1}{2G_z}\sigma_{xz}
\end{aligned}\right\} \tag{27}$$

where E, G, and ν are the Young's modulus, shear modulus, and Poisson's ratio, respectively, in any direction on the x-y plane, E_z and G_z are those for the z-direction. ν_x is the Poisson's ratio corresponding to the contraction in the x-direction for a unit elongation in the z-direction. Likewise ν_z is the Poisson's ratio representing the contraction in z-direction for a unit elongation in the x-direction. These elastic constants are related to each other by symmetry relations, since the strain energy must be a positive definite form. Thus:

$$\nu_x/E_z = \nu_z/E, \qquad G = E/2(1 + \nu) \tag{28}$$

To obtain the plate stress-strain relations we now select a particular type of orthotropy which is consistent with equations (28):

$$\nu_x = e^2\alpha\nu_z \quad , \quad E_z = e^2\alpha E = e^2E' \; , \; G_z = e^2\gamma G = e^2G' \tag{29}$$

Here, α and γ are positive numbers, and ν_z, E and G are independent of e.

We solve components of stress tensor from equation (27) and use equations (29), hence:

$$\left.\begin{aligned}
\sigma_{xx} &= \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy}) + O(e^4), \; \sigma_{xy} = 2G\varepsilon_{xy} + O(e^4) \\
\sigma_{yy} &= \frac{E}{1-\nu^2}(\varepsilon_{yy} + \nu\varepsilon_{xx}) + O(e^4), \; \sigma_{yz} = 2e^2G'\varepsilon_{yz} + O(e^5) \\
\sigma_{zz} &= \frac{e^2E'}{1-\nu}[\nu_z(\varepsilon_{xx} + \varepsilon_{yy}) + (1-\nu)\varepsilon_z] + O(e^6), \\
\sigma_{zx} &= 2e^2G'\varepsilon_{zx} + O(e^5)
\end{aligned}\right\} \tag{30}$$

We thus find that the components of the stress tensor are of the following orders:

$$(\sigma_{xx},\sigma_{yy},\sigma_{xy}) = O(e^2), \; (\sigma_{yz},\sigma_{zx}) = O(e^3), \; \sigma_{zz} = O(e^4) \tag{31}$$

This result agrees with the physical situation that in a shear deformable media shear modulus G_z is small, and in a transverse stress deformable media E_z is small. The plate stress-strain relations (30) can be so adjusted as to include a special type of internal damping by replacing E, G, and G' by $\underset{*}{E}$, $\underset{*}{G}$ and $\underset{*}{G'}$, which are defined by:

$$\underset{*}{E} = E + e* \; , \quad \underset{*}{G} = G + g* \; , \quad \underset{*}{G'} = G' + g'* \tag{32}$$

where E, G, and G' are the usual elastic constants, and star(*)

multiplication is defined by equation (18). We note here, however, that in this particular internal damping, $\underset{*}{\lambda}$ is a constant multiple of $\underset{*}{\mu}$. As a matter of fact:

$$\underset{*}{\lambda} = \frac{2\nu}{1-2\nu}\underset{*}{\mu} \quad , \quad \underset{*}{\mu} = \frac{\underset{*}{E}}{2(1+\nu)} = \underset{*}{G} \tag{33}$$

We shall, however, keep $\underset{*}{G'}$ as an independent operator throughout the present paper.

The simplification introduced by equation (33) is not necessary for the analysis. The whole analysis can be carried out without this assumption. However, for the sake of simplicity we shall use the assumption contained in equations (33). Thus, we have two functional operators $\underset{*}{E}$ and $\underset{*}{G'}$ instead of the three representing the hereditary characteristics of the medium. We note that many important practical classes of problems are contained in the present type of internal damping.

In the present study we shall use the stress-strain relations (19) subject to (33), which are valid for three-dimensional elastic media with internal damping. We shall, then, indicate the terms which should be excluded from the resulting expressions, whenever the foregoing type of anisotropy is assumed, in order to obtain the formulas which would follow if the plate stress-strain relations were used to begin with.

5. Internal Force and Moment Components

In a plate theory, integrals of the stress components, as well as their moments about the x and y axes, play an important role. Thus, we define normal force components (N_x, N_y, N_z), plane shear force component N_{xy}, transverse shear components (Q_x, Q_y), bending moment components (M_x, M_y, M_{yx}), and body force and moment components (P_x, P_y, P_z) and (G_x, G_y, G_z) by the following equations (Fig. 1):

$$
\left.
\begin{aligned}
&(N_x, N_y, N_z, N_{xy}) = \int_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy})\, dz \\
&(Q_x, Q_y) = \int_{h/2}^{h/2} (\sigma_{xz}, \sigma_{yz})\, dz \\
&(M_x, M_y, M_{yx}) = \int_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{yy}, \sigma_{yx}) z\, dz \\
&M_{xy} = -M_{yx} \quad , \qquad N_{xy} = -N_{yx} \\
&(a^2 e^3) \quad (P_x, P_y, P_z) = \int_{-h/2}^{h/2} (F_x, F_y, F_z)\, dz \\
&(a^3 e^4/12) \quad (G_x, G_y, G_z) = \int_{-h/2}^{h/2} (F_x, F_y, F_z) z\, dz
\end{aligned}
\right\} \qquad (34)
$$

where F_x, F_y, F_z are the body force components per unit mass.

Combining equations (5), (19) and (34) we obtain:

$$\left.\begin{aligned}
\sigma_{xx} &= (N_x/h) + (12/h^2)\,\zeta M_x + O(e^4)\\
\sigma_{yy} &= (N_y/h) + (12/h^2)\,\zeta M_y + O(e^4)\\
\sigma_{zz} &= e^2(q + 2p\zeta) + O(e^4)\\
\sigma_{xy} &= (N_{xy}/h) + (12/h^2)\,\zeta M_{yx} + O(e^4)\\
\sigma_{yz} &= (Q_y/h) + O(e^5)\\
\sigma_{xz} &= (Q_x/h) + O(e^5)
\end{aligned}\right\} \quad (35)$$

where:

$$\left.\begin{aligned}
2p &= [\sigma_{zz}(x,y,h/2,t) - \sigma_{zz}(x,y,-h/2,t)]/e^2 = \lambda_*(u_{1,\xi}\\
&\quad + v_{1,\eta} + 2w_2) + 4\mu_* w_2\\
2q &= [\sigma_{zz}(x,y,h/2,t) + \sigma_{zz}(x,y,-h/2,t)]/e^2 = 2\lambda_*\\
&\quad [u_{o,\xi} + v_{o,\eta} + \tfrac{1}{2}w_{o,\xi}{}^2 + \tfrac{1}{2}w_{o,\eta}{}^2 + \tfrac{1}{2}(u_1{}^2 + v_1{}^2)]\\
&\quad + 2\mu_*(u_1{}^2 + v_1{}^2)
\end{aligned}\right\} \quad (36)$$

If we eliminate w_2 and $\frac{1}{2}(u_1{}^2 + v_1{}^2)$ between equations (34), after using equations (19), and (36) we obtain:

$$\left.\begin{aligned}
N_x/h &= \frac{e^2}{1-\nu^2} E_*[u_{o,\xi} + \tfrac{1}{2}w_{o,\xi}{}^2 + \nu(v_{o,\eta} + \tfrac{1}{2}w_{o,\eta}{}^2)] + \frac{\nu}{1-\nu}e^2 q\\
N_y/h &= \frac{e^2}{1-\nu^2} E_*[v_{o,\eta} + \tfrac{1}{2}w_{o,\eta}{}^2 + \nu(u_{o,\xi} + \tfrac{1}{2}w_{o,\xi}{}^2)] + \frac{\nu}{1-\nu}e^2 q\\
N_z/h &= e^2 q\\
N_{xy}/h &= \frac{e^2}{2(1+\nu)} E_*(u_{o,\eta} + v_{o,\xi} + w_{o,\xi}w_{o,\eta})\\
Q_x/h &= \kappa e^3 G'_*(u_1 + w_{o,\xi})\\
Q_y/h &= \kappa e^3 G'_*(v_1 + w_{o,\eta})
\end{aligned}\right\} \quad (37)$$

and:

$$\left.\begin{aligned}
12M_x/h^2 &= \frac{e^2}{1-\nu^2}\frac{E}{*}(u_{1,\xi} + \nu v_{1,\eta}) + \frac{2\nu e^2}{1-\nu}\,p \\
12M_y/h^2 &= \frac{e^2}{1-\nu^2}\frac{E}{*}(v_{1,\eta} + \nu u_{1,\xi}) + \frac{2\nu e^2}{1-\nu}\,p \\
12M_{yx}/h^2 &= \frac{e^2}{2(1+\nu)}\frac{E}{*}(u_{1,\eta} + v_{1,\xi})
\end{aligned}\right\} \quad (38)$$

We note that the terms containing p and q in equations (37) and (38) appear because we used the stress-strain relations (19), which are given for three-dimensional isotropic media. Instead, if we use the plate stress-strain relations (30), we find that these terms disappear from the expressions of N_x, N_y, M_x, and M_y. since in this case $\sigma_{zz} = O(e^4)$ hence p and q become of order e^4. See also [8]. In this case, obviously $N_z = O(e^4)$, but it must be taken into account, as there is no lower order term in the expression of N_z/h.

The introduction of a constant κ into the expressions of Q_x and Q_y is due to the fact that transverse shear stress distribution across the thickness is not constant. It is known, however, that ε_{yz} and ε_{zx} are even functions of z. Hence integrations across the thickness introduce such a constant κ. Such a constant was first introduced by Mindlin [4], and sucessfully replaces κ' of Timoshenko's beam theory, [21] and 5/6 of the Reissner's plate theory [2]. Mindlin has also suggested a formula for κ in his paper.

It may be of interest also to note that we reached the present combination of $\underset{*}{E}$ and ν in the first terms of equation (37) and (38) which are identical to those of the classical theory, not by using two-dimensional stress-strain relations, but by using the three-dimensional stress-strain relations.

6. Equations of Motion

The equations of motion of an elastic body referred to the coordinates in the deformed stage, are sufficient to determine the exact plate equations of motion in a large deflection theory. Integration across the thickness in the deformed position makes it possible to introduce membrane forces and the plate shear-force and bending moment components systematically. In the present theory, the plate thickness is assumed to remain constant throughout the motion.

Equations of motion in curvilinear coordinates are [9]:

$$\sigma^{ij}{}_{;j} = \rho(a^i - F^i) \tag{39}$$

where ρ is the mass density per unit volume, F^i are the components of body forces per unit mass and a^i are the components of the acceleration vector. Indices after the semicolon, ; , represents covariant differentiation, i.e:

$$\sigma^{ij}_{;\ell} = \frac{\partial\sigma^{ij}}{\partial x_\ell} + \Gamma^{i}_{\alpha\ell}\,\sigma^{\alpha j} + \Gamma^{j}_{\alpha\ell}\,\sigma^{i\alpha} \tag{40}$$

Here Γ^k_{ij} is the Christoffel symbol of the second kind and is related to the Christoffel symbol of the first kind $\Gamma_{ij,k}$ and the components of the fundamental metric tensor g^{ij} and

g_{ij} of the deformed medium by:

$$\Gamma^k_{ji} = \Gamma^k_{ij} = g^{k\alpha}\Gamma_{ij,\alpha} = \frac{1}{2} g^{k\alpha}\left(\frac{\partial g_{i\alpha}}{\partial x^j} + \frac{\partial g_{j\alpha}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\alpha}\right) \tag{41}$$

Using equations (1), (5) and (41) we find that the components of $\Gamma_{ij,k}$ are:

$$\left.\begin{aligned}
&\Gamma_{11,1} = (e^2/2a)\,(u_{o,\xi} + \tfrac{1}{2}w_{o,\xi}{}^2 + \zeta u_{1,\xi})_{,\xi}\,,\\
&\Gamma_{11,2} = (e^2/2a)\,(v_{o,\xi\xi} + w_{o,\eta}w_{o,\xi\xi} + \zeta v_{1,\xi\xi})\,,\\
&\Gamma_{11,3} = (e/2a)w_{o,\xi\xi}\,,\quad \Gamma_{12,1} = \Gamma_{21,1} = (e^2/2a)(u_{o,\xi}\\
&\qquad + \tfrac{1}{2}w_{o,\xi}{}^2 + \zeta u_{1,\xi})_{,\eta}\\
&\Gamma_{12,2} = \Gamma_{21,2} = (e^2/2a)(v_{o,\eta} + \tfrac{1}{2}w_{o,\eta}{}^2 + \zeta v_{1,\eta})_{,\xi}\,,\\
&\Gamma_{12,3} = \Gamma_{21,3} = (e/2a)w_{o,\xi\eta}\,,\quad \Gamma_{13,1} = \Gamma_{31,1} = (e/2a)u_{1,\xi}\,,\\
&\Gamma_{13,2} = \Gamma_{31,2} = (e/2a)v_{1,\xi}\,,\quad \Gamma_{13,3} = \Gamma_{31,3} = (e^2/2a)(\tfrac{1}{2}u_1{}^2\\
&\qquad + \tfrac{1}{2}v_1{}^2 + 2\zeta w_2)_{,\xi}\\
&\Gamma_{22,1} = (e^2/2a)(u_{o,\eta\eta} + w_{o,\xi}w_{o,\eta\eta} + \zeta u_{1,\eta\eta})\,,\\
&\Gamma_{22,2} = (e^2/2a)(v_{o,\eta} + \tfrac{1}{2}w_{o,\eta}{}^2 + \zeta v_{1,\eta})_{,\eta}\,,\\
&\Gamma_{22,3} = (e/2a)w_{o,\eta\eta}\,,\quad \Gamma_{23,1} = \Gamma_{32,1} = (e/2a)u_{1,\eta}\,,\\
&\Gamma_{23,2} = \Gamma_{32,2} = (e/2a)v_{1,\eta}\,,\quad \Gamma_{23,3} = \Gamma_{32,3} = (e^2/2a)(\tfrac{1}{2}u_1{}^2 + \tfrac{1}{2}v_1{}^2\\
&\qquad + 2\zeta w_2)_{,\eta}\,,\quad \Gamma_{33,1} = -\Gamma_{13,3}\,,\quad \Gamma_{33,2} = -\Gamma_{23,3}\,,\quad \Gamma_{33,3} = (e/a)w_2
\end{aligned}\right\} \tag{42}$$

The terms of the lowest order in $\sigma^{ij}{}_{;j}$ are:

$$\left.\begin{aligned} &\sigma^{1j}{}_{;j} = \frac{\partial\sigma^{1j}}{\partial x^j}, \quad \sigma^{2j}{}_{;j} = \frac{\partial\sigma^{2j}}{\partial x^j}, \\ &\sigma^{3j}{}_{;j} = \frac{\partial\sigma^{3j}}{\partial x^j} + \Gamma^3_{11}\sigma^{11} + 2\Gamma^3_{12}\sigma^{12} + \Gamma^3_{22}\sigma^{22} \end{aligned}\right\} \quad (43)$$

Thus, in the present theory, we only need to retain Γ^3_{11}, Γ^3_{12} and Γ^3_{22}. A more general theory will undoubtedly contain more of these coefficients. Therefore, equations (42) may be useful for that purpose.

Using equations (1), (41) and (42) we find that:

$$\left.\begin{aligned} \Gamma^3_{11} &= \Gamma_{11,3} + O(e^3) = (e/2a)w_{o,\xi\xi} + O(e^3) \\ \Gamma^3_{12} &= \Gamma_{12,3} + O(e^3) = (e/2a)w_{o,\xi\eta} + O(e^3) \\ \Gamma^3_{22} &= \Gamma_{22,3} + O(e^3) = (e/2a)w_{o,\eta\eta} + O(e^3) \end{aligned}\right\} \quad (44)$$

Components a^i of the acceleration vector in the cartesian coordinates x^i are given by:

$$a^i = \frac{\partial^2 u^i}{\partial t^2} \quad (45)$$

Hence by equations (3)and (45):

$$\left.\begin{aligned} a^1 &= a_x = a\,e^2(\ddot{u}_o + \zeta\ddot{u}_1) + O(e^4) \\ a^2 &= a_y = a\,e^2(\ddot{v}_o + \zeta\ddot{v}_1) + O(e^4) \\ a^3 &= a_z = a\,e\,\ddot{w}_o + O(e^3) \end{aligned}\right\} \quad (46)$$

Microscopic equations of motion (39) thus become:

$$\left.\begin{aligned}
&\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} = \rho(a_x - F_x)\\
&\frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z} = \rho(a_y - F_y)\\
&\frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} + \Gamma^3_{11}\sigma_{xx} + 2\Gamma^3_{12}\sigma_{xy} + \Gamma^3_{22}\sigma_{yy} = \rho(a_z - F_z)
\end{aligned}\right\} \quad (47)$$

Mass density ρ is given by

$$\rho = \rho_o(1 - 2\theta + 4\theta_2 - 8\theta_3)^{1/2} = \rho_o[1 - O(e^2)] \quad (48)$$

where θ, θ_2, θ_3 are the strain invariants of which $\theta = \varepsilon_i^{\,i} = O(e^2)$ is the lowest order. Hence in the present theory $\rho = \rho_o$, that is the change in mass density during the deformation is negligible. The force and the moment equilibrium of a plate element with thickness h, cut off from the plate by drawing normals to the deformed median plane, is commonly used [5] to reduce the microscopic equilibrium conditions to macroscopic equilibrium equations. This, of course, represents an approximation which replaces the correct equilbrium condition, namely, that moments of all order across the thickness should be zero. The foregoing practice, however, only insures zero moments of the first and zero orders. This common practice is, however, in accord with the present theory. Hence integrating equations (47) across the thickness, we obtain:

$$\left.\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + [\sigma_{xz}]_{-h/2}^{h/2} = \int_{-h/2}^{h/2} \rho(a_x - F_x)\,dz \quad\right\} (49)$$

$$\left.\begin{aligned}
&\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + [\sigma_{yz}]_{-h/2}^{h/2} = \int_{-h/2}^{h/2} \rho (a_y - F_y) dz \\
&\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + [\sigma_{zz}]_{-h/2}^{h/2} + \Gamma_{11}^{3} N_x + 2\Gamma_{12}^{3} N_{xy} + \Gamma_{22}^{3} N_y \\
&\quad = \int_{-h/2}^{h/2} \rho (a_z - F_z) dz
\end{aligned}\right\} \tag{49}$$

Next we multiply equations (47) by z and integrate across the thickness. Hence:

$$\left.\begin{aligned}
&\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} + \frac{h}{2}[\sigma_{xz}(x,y,h/2,t) + \sigma_{xz}(x,y,-h/2,t)] - Q_x \\
&\quad = \int_{-h/2}^{h/2} \rho (a_x - F_x) z\, dz \\
&\frac{\partial M_{yx}}{\partial x} + \frac{\partial M_y}{\partial y} + \frac{h}{2}[\sigma_{yz}(x,y,h/2,t) + \sigma_{yz}(x,y,-h/2,t)] - Q_y \\
&\quad = \int_{-h/2}^{h/2} \rho (a_y - F_y) z\, dz \\
&\frac{\partial}{\partial x} \left(\int_{-h/2}^{h/2} \sigma_{xz}\, z\, dz\right) + \frac{\partial}{\partial y} \left(\int_{-h/2}^{h/2} \sigma_{yz}\, z\, dz\right) + \frac{h}{2}[\sigma_{zz}(x,y,h/2,t) \\
&\quad + \sigma_{zz}(x,y,-h/2,t)] - N_z = \int_{-h/2}^{h/2} \rho (a_z - F_z) z\, dz
\end{aligned}\right\} \tag{50}$$

We assume that the surface shear is zero:

$$\sigma_{xz}(x,y,\pm h/2,t) = \sigma_{yz}(x,y,\pm h/2,t) = 0 \tag{51}$$

Using equations (37), (38), (44), and (46), in equations

(49) and (50) we obtain:

$$
\left.
\begin{aligned}
&\frac{e^3}{1-\nu^2} E_* [u_{o,\xi\xi} + \nu v_{o,\eta\xi} + \tfrac{1}{2} (w_{o,\xi}{}^2 + \nu w_{o,\eta}{}^2)_{,\xi}] \\
&\quad + \frac{\nu}{1-\nu} e^3 q_{,\xi} + \frac{e^3}{2(1+\nu)} E_* [u_{o,\eta\eta} + v_{o,\xi\eta} + (w_{o,\xi} w_{o,\eta})_{,\eta}] \\
&\quad + O(e^5) = \rho a^2 e^3 (\ddot{u}_o - P_x) + O(e^5) \quad , \\
&\frac{e^3}{1-\nu^2} E_* [v_{o,\eta\eta} + \nu u_{o,\xi\eta} + \tfrac{1}{2} (w_{o,\eta}{}^2 + \nu w_{o,\xi}{}^2)_{,\eta}] + \frac{\nu}{1-\nu} e^3 q_{,\eta} \\
&\quad + \frac{e^3}{2(1+\nu)} E_* [u_{o,\xi\eta} + v_{o,\xi\xi} + (w_{o,\xi} w_{o,\eta})_{,\xi}] + O(e^5) \\
&\quad = \rho a^2 e^3 (\ddot{v}_o - P_y) + O(e^5) \\
&\kappa e^4 G_*' (u_{1,\xi} + v_{1,\eta} + w_{o,\xi\xi} + w_{o,\eta\eta}) + 2e^2 p \\
&\quad + \left\{ \frac{e^4}{1-\nu^2} E_* [u_{o,\xi} + \tfrac{1}{2} w_{o,\xi}{}^2 + \nu (v_{o,\eta} + \tfrac{1}{2} w_{o,\eta}{}^2)] \right. \\
&\quad \left. + \frac{\nu}{1-\nu} e^4 q \right\} w_{o,\xi\xi} + \left[\frac{e^4}{(1+\nu)} E_* (u_{o,\eta} + v_{o,\xi} + w_{o,\xi} w_{o,\eta}) \right] \\
&\quad w_{o,\xi\eta} + \left\{ \frac{e^4}{1-\nu^2} E_* [v_{o,\eta} + \tfrac{1}{2} w_{o,\eta}{}^2 + \nu (u_{o,\xi} + \tfrac{1}{2} w_{o,\xi}{}^2)] \right. \\
&\quad \left. + \frac{\nu}{1-\nu} e^4 q \right\} w_{o,\eta\eta} + O(e^6) = \rho a^2 e^2 \ddot{w}_o + O(e^4)
\end{aligned}
\right\} \quad (52)
$$

$$
\begin{aligned}
&\frac{e^3}{12(1-\nu^2)} E_* (u_{1,\xi\xi} + \nu v_{1,\xi\eta}) + \frac{\nu e^3}{6(1-\nu)} p_{,\xi} + \frac{e^3}{24(1+\nu)} E_* \\
&\quad (u_{1,\eta\eta} + v_{1,\xi\eta}) - \kappa e^3 G_*' (u_1 + w_{o,\xi}) + O(e^5) \\
&\quad = \frac{1}{12} \rho a^2 e^3 (\ddot{u}_1 - G_x) + O(e^6) \quad ,
\end{aligned}
$$

$$\left.\begin{aligned}
&\frac{e^3}{12(1-\nu^2)} E_* (v_{1,\eta\eta} + \nu u_{1,\xi\eta}) + \frac{\nu e^3}{6(1-\nu)} p_{,\eta} \\
&\quad + \frac{e^3}{24(1+\nu)} E_* (u_{1,\eta\xi} + v_{1,\xi\xi}) - \kappa e^3 G_*'(v_1 + w_{o,\eta}) \\
&\quad + O(e^5) = \frac{1}{12} \rho a^2 e^3 (\ddot{v}_1 - G_y) + O(e^6) , \\
&\frac{e^5}{24(1+\nu)} E_* [(2w_2 w_{o,\xi})_{,\xi} + (2w_2 w_{o,\eta})_{,\eta}] = 0
\end{aligned}\right\} \quad (53)$$

Equations (52) and (53) are the equations of motion.

We note that to obtain these equations we used three-dimensional stress-strain relations. These equations can be reduced further if we take e^2q and e^2p for q and p respectively, thus leading to a theory in which two-dimensional plate stress-strain relations are employed. In this case, all terms involving q and p drop out except the $2e^2p$ term of the last of equation (53). We also note that the inertia term in the latter equation is of $O(e^2)$. This point will be cleared when we used perturbation in time t later. As the last of equation (51) is of $O(e^5)$, we ignore this equation in the present theory.

7. Reduction of the Equations of Motion

The third equation of (52) may be written in the following form:

$$\left.\begin{aligned}
&\kappa e^2 \underset{*}{G'}\,(u_{1,\xi} + v_{1,\eta}) = -\kappa e^2 \underset{*}{G'}\Delta w_o + \rho a^2 \ddot{w}_o - 2p - e^2 N, \\
&e^2 h N = N_x w_{o,\xi\xi} + 2N_{xy} w_{o,\xi\eta} + N_y w_{o,\eta\eta}, \quad \Delta = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}
\end{aligned}\right\} \quad (54)$$

From the first and the second of equations (53) v_1 and u_1 are eliminated with the use of equation (54). Hence:

$$\left.\begin{aligned}
&e^3 \kappa \underset{*}{G'}\Big[\frac{1}{24(1+\nu)} \underset{*}{E}\, \Delta u_1 - \kappa \underset{*}{G'}\, u_1 - \frac{1}{12}\rho a^2 (\ddot{u}_1 - G_x) - \kappa \underset{*}{G'} w_{o,\xi}\Big] \\
&\quad + \frac{e^3}{24(1-\nu)} \underset{*}{E}\big[(\rho\, a^2/e^2)\ddot{w}_{o,\xi} - \kappa \underset{*}{G'}\Delta w_{o,\xi} - N_{,\xi}\big] \\
&\quad + (2\nu e^2 \kappa \underset{*}{G'} - \underset{*}{E})\; e p_{,\xi}/12(1-\nu) = 0 \qquad , \\
&e^3 \kappa \underset{*}{G'}\Big[\frac{1}{24(1+\nu)} \underset{*}{E}\, \Delta v_1 - \kappa \underset{*}{G'} v_1 - \frac{1}{12}\rho a^2 (\ddot{v}_1 - G_y) - \kappa \underset{*}{G'} w_{o,\eta}\Big] \\
&\quad + \frac{e^3}{24(1-\nu)} \underset{*}{E}\big[(\rho\, a^2/e^2)\ddot{w}_{o,\eta} - \kappa \underset{*}{G'}\Delta w_{o,\eta} - N_{,\eta}\big] \\
&\quad + \big[2\nu e^2 \kappa \underset{*}{G'} - \underset{*}{E}\big]\; e p_{,\eta}/12(1-\nu) = 0
\end{aligned}\right\} \quad (55)$$

where we used the following properties of the operators $\underset{*}{E}$ and $\underset{*}{G'}$:

$$\left.\begin{aligned}
&\underset{*}{E}\,\underset{*}{G'} = \underset{*}{G'}\,\underset{*}{E}\,, \quad \frac{\partial}{\partial\xi}\big[\underset{*}{E}\,\phi(\xi,\eta,t)\big] = \underset{*}{E}\,\frac{\partial\phi}{\partial\xi}\,, \\
&\frac{\partial}{\partial\eta}\big[\underset{*}{E}\,\phi(\xi,\eta,t)\big] = \underset{*}{E}\,\frac{\partial\phi}{\partial\eta}
\end{aligned}\right\} \quad (56)$$

which can be proven easily from the defining equations (32). We can also, by part integration, prove that:

$$\left.\begin{aligned}(\underset{*}{E}\,\phi)_{,t} - \underset{*}{E}\,(\phi_{,t}) &= \varepsilon\,(t)\phi(\xi,\eta,0)\\(\underset{*}{E}\,\phi)_{,tt} - \underset{*}{E}(\phi_{,tt}) &= \dot{\varepsilon}(t)\phi(\xi,\eta,0) + \varepsilon(t)\dot{\phi}(\xi,\eta,o)\end{aligned}\right\}\quad(57)$$

These identities will be used in the following analysis.

Differentiating the first of equations (55) with respect to ξ and the second with respect to η, adding the results to each other and using equations (54), (56) and (57) we obtain:

$$\left.\begin{aligned}&-\kappa\, e^2\underset{*}{G'}\underset{*}{D}\,\Delta^2 w_o + \rho a^2(\underset{*}{D} + \frac{h^3}{12}\kappa e^2\underset{*}{G'})\Delta\ddot{w}_o - \rho a^4 h\,\kappa e^2\underset{*}{G'}\ddot{w}_o\\&-\frac{\rho^2 h^3 a^4}{12}\ddddot{w}_o + 2[-\underset{*}{D} + \frac{\nu h^3}{12(1-\nu)}\,\kappa\, e^2\underset{*}{G'}]\Delta p + 2h^3\kappa\,\underset{*}{G'}p\\&+\frac{\rho h^3 a^2}{6}\ddot{p} - e^2\underset{*}{D}\,\Delta N + h^3\,\kappa e^2\underset{*}{G'}\,N + \frac{\rho h^5}{12}\,\ddot{N}\\&+\frac{\rho a^2 h^3}{12}\,\kappa\, e^2\underset{*}{G'}\,(G_{x,\xi} + G_{y,\eta}) + \kappa\frac{\rho h^3}{12}\,[\gamma,\ \Delta w_o\\&+ u_{1,\xi} + v_{1,\eta}] = 0\end{aligned}\right\}\quad(58)$$

where:

$$\left.\begin{aligned}\underset{*}{D} &= \frac{h^3}{12(1-\nu^2)}\underset{*}{E}\\[\gamma,\ \phi] &= \dot{\gamma}(t)\phi(\xi,\eta,o) + \gamma(t)\dot{\phi}(\xi,\eta,o)\end{aligned}\right\}\quad(59)$$

Here the operator $\underset{*}{D}$ takes the place of the bending rigidity of the plate theory without internal damping. $[\gamma,\ldots]$ is known when the initial conditions are given. Equations (54), (55) and (58) are the equations of motion for rotatory and transverse motion. In case of zero internal damping, $[\gamma,\ldots]\equiv 0$

and star indexed quantities G', E and D must be replaced by $\overset{*}{G}'$, $\overset{*}{E}$ and $\overset{*}{D}$. Hence, we can divide these equations throughout by $\kappa e^2 G'$ leading to plate equations with no internal damping. This is done only for equation (58) leading to:

$$\left.\begin{aligned} &-D\Delta^2 w_o + \left(\frac{\rho a^2 h^3}{12} + \frac{D\rho a^2}{\kappa e^2 G'}\right)\Delta \ddot{w}_o - \rho a^4 h\, \ddot{w}_o \\ &\quad -(\rho^2 h^3 a^4/12\kappa e^2 G')\dddot{\ddot{w}}_o + 2\left[-\frac{D}{\kappa e^2 G'} + \frac{\nu h^3}{12(1-\nu)}\right]\Delta p \\ &\quad + (2h^3/e^2)p + (\rho h^3 a^2/12\kappa e^2 G')\, 2\ddot{p} - (D/\kappa G')\Delta N \\ &\quad + h^3 N + (\rho h^5/12\,\kappa e^2 G')\ddot{N} + (\rho a^2 h^3/12)(G_{x,\xi} + G_{y,\eta}) = 0 \\ &D = Eh^3/12(1-\nu^2) \end{aligned}\right\} \quad (60)$$

Equation (60) takes the place of the classical plate equation:

$$-D\,\Delta^2 w_o - \rho a^4 h\, \ddot{w}_o + 2a^2 h\, p + h^3 N = 0 \qquad (61)$$

which are four terms of equation (60). The terms containing G' represent the effect of shear deformation and the terms which are differentiated with respect to time and which contain $(\rho h^3/12)$ as a coefficient represent the effect of rotatory inertia. The effect of the large deformation is contained in the expressions of N, since N_x, N_y, and N_{xy}, given by equation (37), are multiplied by various derivatives of w in the expression (54) of N leading to a non-linear equation. Body moment components are taken into account with terms $G_{x,\xi}$ and $G_{y,\eta}$. The term $[(\nu h^3/12(1-\nu)]\Delta p$ is not encountered in the classical theory. This is due to the transverse stress component which is ignored in the classical plate theory. The foregoing term is the moment of the plane forces due to contraction in the x and y direction in the presence of surface pressure p. Hence: In order to obtain classical plate equation, we must let $G' \rightarrow \infty$, $\rho h^2/12 \rightarrow o$, while $\rho a^4 h$ = finite value and $p \rightarrow e^2 p$.

Axial displacements u_o and v_o satisfy the first and second of equations (52) which may be transformed into simpler forms:

$$\underset{*}{E}\,\Delta\omega_o - 2(1+\nu)\rho a^2\ddot{\omega}_o = \underset{*}{E}\,(w_{o,\eta}\Delta w_{o,\xi} - w_{o,\xi}\,\Delta w_{o,\eta}) + 2\rho a^2(1+\nu)\,[-P_{x,\eta} + P_{y,\xi}] \tag{62}$$

$$\underset{*}{E}\,\Delta\theta_o - \rho a^2(1-\nu^2)\ddot{\theta}_o = -\underset{*}{E}\,[w_{o,\xi}\Delta w_{o,\xi} + w_{o,\eta}\Delta w_{o,\eta} + (\Delta w_o)^2 + (1+\nu)(w_{o,\xi\eta}^{\;2} - w_{o,\xi\xi}w_{o,\eta\eta})] - \nu(1+\nu)\Delta q + \rho a^2(1-\nu^2)\,[-P_{x,\xi} - P_{y,\eta}] \tag{63}$$

where ω_o and θ_o are the rotation and the dilatation defined by:

$$\theta_o = u_{o,\xi} + v_{o,\eta}\,, \qquad \omega_o = u_{o,\eta} - v_{o,\xi} \tag{64}$$

Between u_o, v_o, and θ_o, ω_o dual relations exist which are expressed by equations (64) and (65);

$$\Delta u_o = \theta_{o,\xi} + \omega_{o,\eta} \qquad \Delta v_o = \theta_{o,\eta} - \omega_{o,\xi} \tag{65}$$

We note that when the right sides of equations (62) and (63) are taken as zero and $\underset{*}{E}$ is replaced by E, we obtain the plane rotational and dilational wave equations of the small deflection theory. The right hand sides of these terms represent the effects of finite deformation, body forces and the effect of plane contraction due to a load q.

In application, boundary and initial conditions are usually given in terms of planar displacements u_o and v_o, hence it is useful to express differential equations in terms of u_o and v_o. This is done by first expressing the first two of equations (52) in terms of u_o, $\omega_{o,\eta}$ and v_o, $\omega_{o,\xi}$, and then eliminating ω_o with the use of equations (62). Hence:

$$\left.\begin{aligned} \Delta'\Delta^* u_o + \nu(1+\nu)\Delta' q_{,\xi} + (1+\nu)^2\rho a^2[\varepsilon,\omega_{o,\eta}] &= A^* \\ \Delta'\Delta^* v_o + \nu(1+\nu)\Delta' q_{,\eta} - (1+\nu)^2\rho a^2[\varepsilon,\omega_{o,\xi}] &= B^* \end{aligned}\right\} \quad (66)$$

where:

$$\left.\begin{aligned} \Delta^* f &= E_*\Delta f - \rho a^2(1-\nu^2)\ddot{f} \\ \Delta' f &= E_*\Delta f - 2\rho a^2(1+\nu)\ddot{f} \\ A^* &= -\tfrac{1}{2}\Delta' E_*[(w_{o,\xi}{}^2 + \nu w_{o,\eta}{}^2)_{,\xi} + (1-\nu)(w_{o,\xi}w_{o,\eta})_{,\eta}] \\ &\quad + \tfrac{1+\nu}{2} E_* E_*[w_{o,\eta}\Delta w_{o,\xi} - w_{o,\xi}\Delta w_{o,\eta}]_{,\eta} - \rho a^2(1-\nu^2)\Delta' P_x \\ &\quad + \rho a^2(1+\nu)^2 E_*[-P_{x,\eta} + P_{y,\xi}]_{,\eta} \end{aligned}\right\} \quad (67)$$

B^* is obtained from A^* by interchanging x,ξ and y,η, respectively.

Equations (66) are the equations of extensional vibrations of visco-elastic plates including the effect of finite transverse oscillations. It is interesting to note that the terms containing transverse deflection w_o are completely separated from the axial displacements u_o and v_o. Thus no coupling effect is introduced. The coupling effect between u_o and v_o is introduced only with the brackets, $[\varepsilon, \omega_{o,\eta}]$ and $[\varepsilon, \omega_{o,\xi}]$ which are the results of initial conditions. Moreover, this coupling effect is introduced with the rotation ω_o. These terms disappear for purely elastic media where memory function $\varepsilon \equiv o$.

Equations (66) are linear in u_o and v_o. Consequently, once we obtain w_o, the right hand members A^* and B^* will be known. Therefore, the extensional vibration problem will be reduced to solving the inhomogeneous linear partial differential equations (66).

For elastic plates with zero internal damping equations (66) can be written as:

$$\left.\begin{aligned} &\Delta^2 u_o - (1+\nu)(3-\nu)E^{-1}\rho a^2 \Delta \ddot{u}_o + 2(1+\nu)(1-\nu^2)E^{-2}\rho^2 a^4 \ddddot{u}_o \\ &\qquad + \nu(1+\nu)E^{-1}\Delta q_{,\xi} - 2\nu(1+\nu)^2 E^{-1}\rho a^2 \ddot{q}_{,\xi} = A\,E^{-2} \\ &\Delta^2 v_o - (1+\nu)(3-\nu)E^{-1}\rho a^2 \Delta \ddot{v}_o + 2(1+\nu)(1-\nu^2)E^{-2}\rho^2 a^4 \ddddot{v}_o \\ &\qquad + \nu(1+\nu)E^{-1}\Delta q_{,\eta} - 2\nu(1+\nu)^2 E^{-1}\rho a^2 \ddot{q}_{,\eta} = B\,E^{-2} \end{aligned}\right\} \quad (68)$$

where A and B are obtained from A^* and B^* by taking E for E_*. Equations (68) are the generalizations of the equations of extensional vibrations of the classical theory. In the classical theory terms containing A, B, and q are ignored.

8. Perturbation in Time

The above differential equations are extremely complicated and do not lend themselves easily to integration. Simplification can be made, however, if some additional assumptions are made. For example, in the application, plates have finite boundaries. The stress or deformation wave in the plane of the plate has a very large wave velocity. Consequently, a few milliseconds after the motion begins, the stress wave will be felt everywhere on the plate. Thus, in problems where the transient state in the planar wave propagation is unimportant, a great deal of simplification can be effected by making perturbation in time. Also assume that the body forces and moments are of $O(e^4)$. Hence let:

$$\tau = et \tag{69}$$

The first two of equation (49) can then be reduced to:

$$N_{x,\xi} + N_{xy,\eta} = 0 \text{ , } N_{xy,\xi} + N_{y,\eta} = 0 \tag{70}$$

since all terms in equation (49) are of order e^3, while inertia terms are of order e^5, (See equations (52)). Equations (70) are satisfied by the Airy stress function $F_o(\xi,\eta,t)$:

$$N_x = e^2 F_{o,\eta\eta} \text{ , } N_{xy} = -e^2 F_{o,\xi\eta} \text{ , } N_y = e^2 F_{o,\xi\xi} \tag{71}$$

u_o and v_o contained in the expressions of N_x, N_y and N_{xy} given by equations(37), can be eliminated, thus leading to compatibility equations:

$$(N_x - \nu N_y)_{,\eta\eta} + (N_y - \nu N_x)_{,\xi\xi} - 2(1+\nu)N_{xy,\xi\eta} = 0 \tag{72}$$

Substitution of equation (71) into (72) gives:

$$\Delta^2 F = \nu h \Delta q + hE_*(w_{o,\xi\eta}{}^2 - w_{o,\xi\xi} w_{o,\eta\eta}) \tag{73}$$

The equation for the transverse vibration is obtained by combining equations (58) and (71). The only change in equation (58) will be in the expression of N, which takes the form:

$$hN = F_{o,\eta\eta} w_{o,\xi\xi} - 2F_{o,\xi\eta} w_{o,\xi\eta} + F_{o,\xi\xi} w_{o,\eta\eta} \tag{74}$$

Of course, we also set $G_x \equiv G_y = 0$, and transform the terms containing derivatives with respect to time to derivatives with respect to τ, by using equation (69). Equation (58) with N given by (74) and equation (73), are solved simultaneously to determine F_o and w_o. Once this is done we can then use the first two of equations (37)or (66) to determine u_o and v_o since N_x, N_y and N_{xy} will be known by equation (71). Deformation components u_1 and v_1 can now be solved from equation (55)

We note that equation (73), without the term $\nu h \Delta q$, and equation (58 with all terms and the internal damping taken as zero, except the first and the ninth terms, gives the large static deflection equation given by the Föppl-Kármán-Timoshenko theory (*). Thus the present theory generalizes this theory to dynamic problems. Moreover, it extends the theory to take into account the shear deformation rotatory inertia, inertial external loading on the surfaces and the boundaries of the plates for visco-elastic plates. Various special cases given below are suggestive in simplifying these equations further for practical problems.

a. Zero internal damping

In this case equation (6C) becomes:

$$-D\Delta^2 w_o + \left(\rho\frac{h^5}{12} + \frac{D\rho a^2}{\kappa G'}\right)\Delta w_{o,\tau\tau} - \rho a^2 h^3 w_{o,\tau\tau}$$
$$-(\rho^2 h^5 a^2/12\kappa G')w_{o,\tau\tau\tau\tau} + 2\left(-\frac{Da^2}{\kappa h^2 G'} + \frac{\nu h^3}{12(1-\nu)}\right)\Delta p + 2ha^2 p$$
$$+(\rho h^3 a^2/12\kappa G')2p_{,\tau\tau} - (D/\kappa G')\Delta N + h^3 N + (\rho h^5/12\kappa G')N_{,\tau\tau} = C \quad (75)$$

where N is given by equation (74).

b. Small deflection and (a)

If we use equation (75), and in all the equations which follow take N as given by the second of equations (54), where N_x,

(*) The theory was given first by Föppl [1].

N_{xy}, and N_y are constants, we obtain small deflection theory.

c. Rotatory inertia omitted and (a)

This means dropping all terms containing $\rho h^3/12$ in equation (75), hence:

$$-D\Delta^2 w_o + (D\rho a^2/\kappa G')\Delta w_{o,\tau\tau} - \rho a^2 h^3 w_{o,\tau\tau}$$

$$+2\left(-\frac{Da^2}{\kappa h^2 G'} + \frac{\nu h^3}{12(1-\nu)}\right)\Delta p + 2ha^2 p - (D/\kappa G')\Delta N + h^3 N = 0 \qquad (76)$$

If we further put $\frac{\partial}{\partial \tau} = 0$ to obtain static plate deflection, we obtain Reissner's equation (70) given in [22], except for the term $\nu h^3/12(1-\nu)$. As explained before, this term does not appear if one starts with the plate stress-strain relations.

d. Shear deformation omitted and (a)

By letting $\kappa G' \to \infty$ in equation (75), we obtain the plate equation in which shear deformation is negligible:

$$-D\Delta^2 w_o + (\rho h^5/12)\Delta w_{o,\tau\tau} - \rho a^2 h^3 w_{o,\tau\tau} + \frac{\nu h^3}{6(1-\nu)}\Delta p$$

$$+ 2ha^2 p + h^3 N = 0 \qquad (77)$$

Further, if we set $\rho h^3/12 = 0$ we obtain the plate equations of Föppl-Kármán-Timoshenko theory except for the term containing $\nu h^3/12(1-\nu)$ whose presence is explained above.

e. Small deflection with axial stresses omitted

By setting $N=0$, equation (76) reduces to the equation given by Uflyand-Mindlin theory [3, 4] except for the term containing $\nu h^3/12(1-\nu)$.

9. Boundary and Initial Conditions

Let n_j be the exterior unit normal to the edge surface Σ and T^i be the stress vectors acting on Σ. Then the boundary conditions on the surface tractions consist of giving T^i on Σ at all times:

$$T^i = \sigma^{ij} n_j \text{ prescibed on } \Sigma \tag{78}$$

For the present theory, there is no distinction between contravariant stress vectors T^i and covariant stress vectors $T_i = \sigma_{ij} n^j$. Let s be the tangent to the boundary curve, C, which is the intersection of the deflected medium plane with Σ. Then n, s, z make a right hand system. Let θ be the angle between n and x and $\frac{\pi}{2} - \theta$ the angle between n and y, Fig. 2. We first transform σ^{ij} components to the n, s, z system. Afterwards, we obtain the plate boundary conditions on the membrane forces by integrating T^i across the thickness from $-h/2$ to $+h/2$. Boundary conditions on the bending moments are obtained by multiplying T^i with z and integrating with respect to z across the thickness. Hence:

$$\left.\begin{aligned} N_n &= \tfrac{1}{2}(N_x + N_y) + \tfrac{1}{2}(N_x - N_y)\cos 2\theta + N_{xy}\sin 2\theta \\ N_{ns} &= \tfrac{1}{2}(N_y - N_x)\sin 2\theta + N_{xy}\cos 2\theta \end{aligned}\right\} \tag{79}$$

$$\left.\begin{aligned} Q_{nz} &= Q_x \cos\theta + Q_y \sin\theta \\ M_n &= \tfrac{1}{2}(M_x + M_y) + \tfrac{1}{2}(M_x - M_y)\cos 2\theta - M_{xy}\sin 2\theta \\ M_{sn} &= \tfrac{1}{2}(M_x - M_y)\sin 2\theta + M_{xy}\cos 2\theta \end{aligned}\right\} \tag{80}$$

We note that like in Reissner's theory, [2] we have three boundary conditions (80) in case of zero membrane forces. In the classical plate theory, of course, the condition on Q_{nz} is absent. In the presence of membrane forces, altogether five boundary conditions must be satisfied.

In case the surface tractions are unknown but some support conditions are given, depending on the type of support, we need to prescribe deformations u_o, v_o, w_o, u_1, v_1 and their derivatives in various directions. These conditions are easy to express, from the meaning of the deformation components. Of course, zero derivatives of these functions in any direction along the boundary express the clamp conditions.

In general, we might have a mixed condition at the edge involving some conditions on traction and some conditions on deformations and their directional derivatives. Altogether, five independent mixed conditions are needed at a part of boundary on which the normal n is continuous and has first order derivatives.

Initial conditions are obtained by prescribing the dependent variables and their time derivatives of first order.

An exhaustive study of the boundary and initial conditions can be obtained by using the variational principle; which, however, seems not worth the trouble.

Finally, we give the expressions of M_x, M_y, and M_{xy}. Hence M_n and M_{sn} can be obtained in terms of w_o by means of equations (80), so that the differential equation for w_o can be treated without reference to the other deformation components. This can be done by combining the last two of equations (37) with equations (38). Hence:

$$\left.\begin{aligned}
\kappa hG_*'M_x &= -(h^2/a^2)\kappa G_*' D_*(w_{o,\xi\xi} + \nu w_{o,\eta\eta}) + (a/h^2)D_*(Q_{x,\xi} \\
&\quad + \nu Q_{y,\eta}) + \frac{\nu}{5(1-\nu)} (h^5/a^2)\kappa G_*' p \\
\kappa hG_*'M_y &= -(h^2/a^2)\kappa G_*' D_*(w_{o,\eta\eta} + \nu w_{o,\xi\xi}) + (a/h^2)D_* \\
&\quad (Q_{y,\eta} + \nu Q_{x,\xi}) + \frac{\nu}{5(1-\nu)} (h^5/a^2)\kappa G_*' p \\
\kappa hG_*'M_{yx} &= -(h^2/a^2)(1-\nu)\kappa G_*' D_* w_{o,\xi\eta} + (a/2h^2) \\
&\quad (1-\nu)D_*(Q_{x,\eta} + Q_{y,\xi})
\end{aligned}\right\} \quad (81)$$

If the plate stress-strain relations were used, the terms containing p would drop out. In case of zero internal damping one can also cancel the coefficients $\kappa hG_*'$.

$$\left.\begin{aligned}
a^2M_x/h &= -D(w_{o,\xi\xi} + \nu w_{o,\eta\eta}) + (a/h)^3(D/\kappa hG')(Q_{x,\xi}+\nu Q_{y,\eta}) \\
&\quad + \nu(1+\nu)(D/E)\ 2p \\
a^2M_y/h &= -D(w_{o,\eta\eta} + \nu w_{o,\xi\xi}) + (a/h)^3(D/\kappa hG')(Q_{y,\eta}+\nu Q_{x,\xi}) \\
&\quad + \nu(1+\nu)(D/E)\ 2p \\
a^2M_{yx}/h &= -(1-\nu)\ Dw_{o,\xi\eta} + \tfrac{1}{2}(a/h)^3(1-\nu)(D/\kappa hG')(Q_{x,\eta}+Q_{y,\xi})
\end{aligned}\right\} \quad (82)$$

In these expressions the terms containing w_o are identical with those of the classical plate theory, while the terms containing Q_x and Q_y are due to shear deformation. Equations (82) are the analogues of Reissner's equation (10) of [2].

10. Strain Energy

The present form of the stress-strain relations (equations 19) implies that the expression of strain energy V is of the following form:

$$V = \int_{(v)} U dv \tag{83}$$

where U is the strain energy density per unit volume.

$$U = \frac{1}{2} \sigma^{ij} \varepsilon_{ij} \tag{84}$$

This form of the strain energy, with stress-strain relations given by equations (19), is identical to that of the infinitesimal theory in the case of zero internal damping. This form is of course valid for infinitesimal strain but large deformation and rotations [23], which is the basis of the present theory.

Combining equations (19), (20), and (84), we obtain:

$$\left.\begin{aligned} U &= U_e + U_d \\ U_e &= \tfrac{1}{2}(\lambda + 2\mu)\theta^2 - 2\mu\theta_2^2 \\ U_d &= \theta L * \theta + 2\varepsilon_{ij} M * \varepsilon_{ij} \end{aligned}\right\} \tag{85}$$

Here U_e is the elastic energy density and U_d, the dissipative energy density. Quantities θ and θ_2 are the first and second strain invariants, of which the first is given by the second of equations (6) and the second is given as:

$$\theta_2 = \varepsilon_2^2\varepsilon_3^3 - \varepsilon_2^3\varepsilon_3^2 + \varepsilon_3^3\varepsilon_1^1 - \varepsilon_3^1\varepsilon_1^3 + \varepsilon_1^1\varepsilon_2^2 - \varepsilon_2^1\varepsilon_1^2 \tag{86}$$

Functions L and M are the memory or internal damping functions.

For the Voight-Sezawa type of internal damping they are given by the second members of equations (26) namely:

$$L = \lambda'\delta_{,t} \qquad M = \mu'\delta_{,t} \tag{87}$$

We now use the expression (5) of ε_{ij} in the expression (85) of U and integrate with respect to ζ across the plate thickness, then use equations (36) to (38). The result is:

$$\left.\begin{aligned}
\bar{U} &= \int_{-h/2}^{h/2} U dz = \bar{U}_N + \bar{U}_M \\
2hE\bar{U}_N &= N_x[N_x - \nu(N_y + N_z)]_e + N_y[N_y - \nu(N_z + N_x)]_e \\
&\quad + N_z[N_z - \nu(N_x + N_y)]_e + 2(1+\nu)N_{xy}[N_{xy}]_e \\
2hE\bar{U}_M &= (6M_x/h^2)[M_x - \nu(M_y + M_z)]_e + (6M_y/h^2)[M_y - \nu(M_z \\
&\quad + M_x)]_e + (6M_z/h^2)[M_z - \nu(M_x + M_y)]_e + 2(1+\nu) \\
&\quad (6M_{yx}/h^2)[M_{yx}]_e + (E/\kappa e^2G')Q_x[Q_x]_e + (E/\kappa e^2G')Q_y[Q_y]_e, \\
6M_z/h^2 &= e^2p = \int_{-h/2}^{h/2} \sigma_{zz}\, z\, dz
\end{aligned}\right\} \tag{88}$$

Here $\bar{U}_N$ and $\bar{U}_M$ are the strain energy densities per unit area due to the membrane force and bending moment components, respectively. Subscript e at the end of a bracket indicates the fact that only the elastic part of the expression inside of the bracket must be used. We note that $\bar{U}_N$ and $\bar{U}_M$ consist of two parts, namely, the elastic energy and dissipative energy due to damping of which the first is obtained by considering the elastic parts of the membrane force and bending moment components which are multiplied by the brackets with subscript e. The dissipative part is the remainder. Introduction of M_z is purely to keep the expression symmetrical.

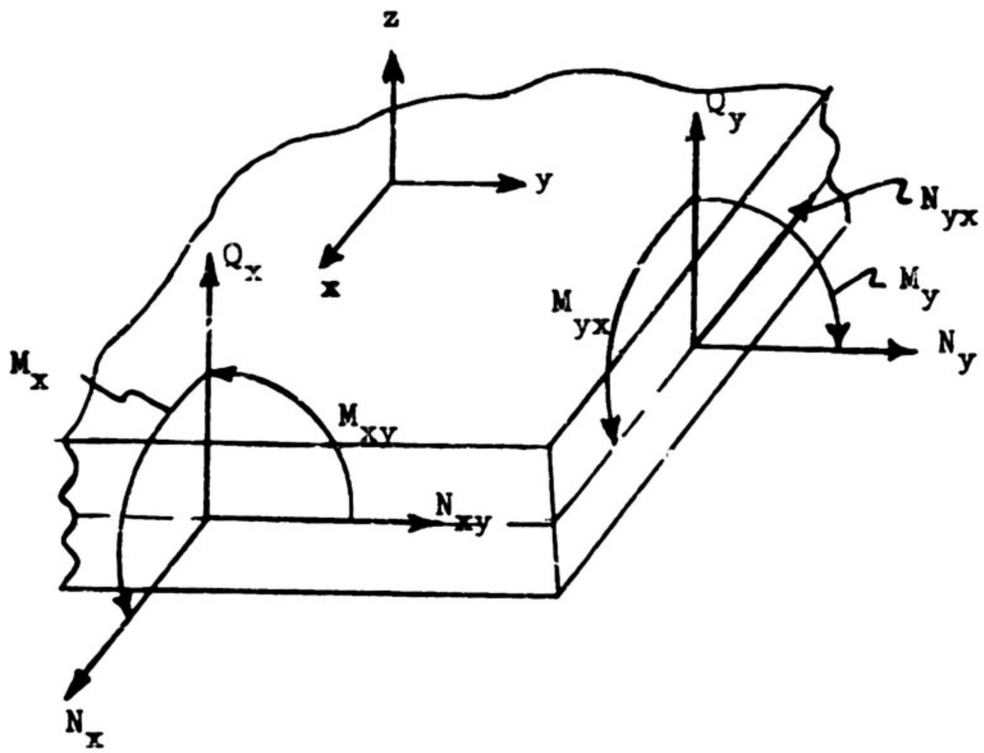

Fig. 1 Plate Force and Moment Components

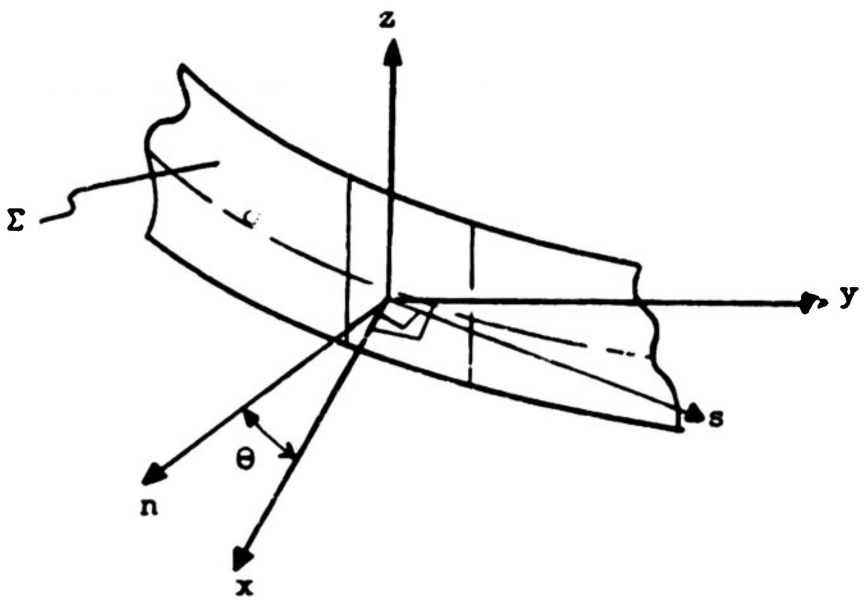

Fig. 2 Orientation of the Coordinates at the Edge Surface

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